Matrix Valued Schrødinger Operators on Local Fields

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The Third International Conference on
p-Adic Mathematical Physics, Moscow 2007

Outline

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Quantum systems with inner symmetries

- Objects of study: Quantum systems with inner symmetries - i.e., with matrix valued Hamiltonians - over a local field.
- Discuss self-adjointness, and obtain a Feynman-Kac formula for the propagators.

Outline

1. Background
   - Description of the problem
   - Previous Work
     - Varadarajan 1997
     - Varadarajan-Weisbart 2007

2. Current work: Matrix Valued Hamiltonians over a local field
   - Setting
   - Results

Varadarajan 1997

- K: A local field.
- D: A division algebra over K.
- W: A finite dimensional vector space over D.
- Hilbert space $\mathcal{H} = L^2(W)$
- Hamiltonian:
  - $H_b = \Delta_b + V$ where $\Delta_b = \mathbf{F} M_b \mathbf{F}^{-1}$.
  - $M_b =$ multiplication by $|x|^b$ for some $b > 0$.
  - $\mathbf{F} =$ Fourier transform on $L^2(W)$.
  - V: (multiplication by) a bounded function on W.

Varadarajan 1997

Fix $t > 0$ and set, for $\xi \in W^*$,

$$\phi_{t,b}(\xi) = \exp(-t|\xi|^b) \, .$$

Then $\phi_{t,b} \in L^m(W^*, d\xi)$ for all $m \geq 1$, and is the Fourier transform of a continuous probability density $p_{t,b}$ on W with $p_{t,b}(ax) = p_{t,b}(x)$ for $x \in W$, $|a| = 1$.

Varadarajan 1997

The $(p_{t,b})_{t>0}$ form a continuous convolution semigroup of probability densities which goes to the Dirac delta measure at 0 when $t \to 0$. Hence for any $x \in W$ one can associate a W-valued separable stochastic process with independent increments, $(X(t))_{t\geq 0}$, with $X(0) = x$, such that for any $t > 0$, $u \geq 0$, $X(t+u) - X(u)$ has the density $p_{t,b}$.

Varadarajan 1997

Skorokhod Spaces:

- $\mathcal{D}([0,\infty):W)$: The space of right continuous functions on $[0,\infty)$ with values in W, having only discontinuities of the first kind.
- For $T>0$, the space $\mathcal{D}([0,T]:W)$ is defined similarly, with an additional requirement about left continuity at T.
- The process $X(t)_{t\geq 0}$ with $X(0)=x$ has paths in the space $\mathcal{D}([0,\infty):W)$ and is concentrated in the subspace of paths taking the value x for $t=0$.
- Consider the processes obtained from $(X(t))_{t\geq 0}$ by conditioning them to go through y at time T. The corresponding probability measures can be defined on the Skorokhod space $\mathcal{D}([0,T]:W)$.

Varadarajan 1997

Theorem

There are unique families of probability measures $\mathbf{P}_x$, $\mathbf{P}^T_{x,y}$, $(x, y \in W)$ *on* $\mathcal{D}([0,\infty) : W)$ *and* $\mathcal{D}([0,T] : W)$*, respectively, continuous with respect to* x *and* (x, y)*, respectively, such that* $\mathbf{P}_x$ *is the probability measure of the* X*-process starting from* x *at time* $t = 0$*, and* $\mathbf{P}^T_{x,y}$ *is the probability measure for the* X*-process that starts from* x *at time* $t = 0$ *and is conditioned to pass through* y *at time* $t = T$*.*

Varadarajan 1997

Theorem (Feynman-Kac)

The operator e^{-tH_b}, $(t > 0)$, is an integral operator in $L^2(W)$ with kernel $K_{t,b}$ on $W \times W$ which is represented by the following integral on the space $\mathcal{D}([0,t] : W)$:

$$K_{t,b}(x : y) = \int_{\omega \in \mathcal{D}([0,t]:W)} \exp\left(-\int_0^t V(\omega(s))ds\right) dP^t_{x,y}(\omega) \cdot p_{t,b}(x - y) \quad (x, y \in W)$$

Varadarajan-Weisbart 2007

- Hilbert space $\mathcal{H} = L^2(\mathbf{R}^d) \otimes \mathbf{C}^m$.
- Potential $V\colon \mathbf{R}^d \to M_m(\mathbf{C})$, continuous. $V(x)$ is a positive matrix for all $x \in \mathbf{R}^d$.
- Hamiltonian $H = -1/2\Delta + V$, where the Laplacian Δ acts only on the left component: $\Delta(f \otimes \phi) = \Delta f \otimes \phi$.
- $U_t = e^{tH}$, $t \geq 0$.

Varadarajan-Weisbart 2007

Feynman-Kac for the scalar case: U_t is an integral operator whose kernel $K_t(x, y)$ is given by:

$$K_t(x,y) = \int_{C([0,t]:\ \mathbf{R}^d)} e^{-\int_0^t V(\omega(s))\, ds} dW_{x,y}^t(\omega) \cdot p_t(x-y)$$

where $C([0,t]:\ \mathbf{R}^d)$ is the space of continuous maps ω from $[0,t]$ to $\mathbf{R}^d$, and $W_{x,y}^t$ is the conditioned Wiener measure that starts from x at time 0 and exits from y at time t.

Varadarajan-Weisbart 2007

In the present matrix valued case the exponential under the integral sign must be replaced by a time-ordered exponential: If a is a function from the reals to the $m \times m$ complex matrices, we define its time-ordered exponential $E(a)$ by:

$$E(a)(t) = 1 + \sum_{n \geq 1} \int_{0<s_n<\cdots<s_1<t} a(s_n) \cdots a(s_1)\, ds_n \cdots ds_1 \, .$$

Varadarajan-Weisbart 2007

$E(a)(t)$ solves the matrix initial value problem:

$$\frac{d}{dt}y = y \cdot a, \quad y(0) = 1\,,$$

or, equivalently, the integral equation:

$$y(t) = 1 + \int_0^t y(t) \cdot a(s)\,ds$$

and coincides with

$$e^{\int_0^t a(s)\,ds}$$

when the $a(t)$'s commute among themselves.

Varadarajan-Weisbart 2007

Theorem (Feynman-Kac for matrix valued Hamiltonians)

Under the stated assumptions e^{-tH} *is an integral operator with kernel* $K_t(x, y)$ *given by*

$$K_t(x,y) = \int_{\omega \in C([0,t]:\,\mathbf{R}^d)} E(-V \circ \omega)(t)\, dW^t_{x,y}(\omega) \cdot p_t(x,y)\,,$$

where

$$p_t(x,y) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x-y|^2}{2t}} \qquad (x, y \in \mathbf{R}^d)$$

Outline

Current work: Setting

- K: A local field.
- W: A finite dimensional vector space over K.
- Hilbert space $\mathcal{H} = L^2(W) \otimes \mathbf{C}^m$.
- Potential $V : W \to M_m(\mathbf{C})$, continuous, $V(x)$ positive for all $x \in W$.
- M_b: multiplication by $|x|^b$ on $L^2(W)$, for some $b > 0$.
- $\mathbf{F}$ = Fourier transform on $L^2(W)$.
- Hamiltonian H_b:
  - $H_b = \Delta_b + V$ where the "Laplacian" $\Delta_b = \mathbf{F} M_b \mathbf{F}^{-1}$ acts only on the first component of $\mathcal{H}$, i.e., on $L^2(W)$.

Outline

Essential self-adjointness of the Hamiltonian

Set

$$S(W) = \{f\colon W \to \mathbf{C}^m;\ f \text{ is locally constant with compact support}\}$$

Proposition

With the above assumptions H_b is essentially self-adjoint on the domain $S(W) \otimes \mathbf{C}^m$.

Proof of essential self-adjointness

Proof (sketch).

We use a technique found in Kochubei's book: For each $x \in W$ let $B(x)$ denote the closed unit ball centered at x, and set $\mathcal{H}(x) = \{f \in \mathcal{H} \colon f \text{ has support in } B(x)\}$. Find a sequence (x_i) in W such that W is the disjoint union of the $B(x_i)$'s. Then $L^2(W) = \oplus_i \mathcal{H}(x_i)$.
Set $D_i = (S(W) \otimes \mathbf{C}^m) \cap \mathcal{H}(x_i)$. Show that H_b leaves each D_i invariant and that H_b is essentially self-adjoint on D_i. Then use that a direct sum of essentially selfadjoint operators is again essentially self-adjoint. □

Feynman-Kac formula for matrix valued Hamiltonians over a local field

Recall from Varadarajan [1997] that the function $e^{-t\|\xi\|^b}$ is the Fourier transform of a probability density function $p_{t,b}$. Further, these give rise to unique families of probability measures $\mathbf{P}_x$, $\mathbf{P}^T_{x,y}$, $(x, y \in W)$ on $\mathcal{D}([0,\infty) : W)$ and $\mathcal{D}([0, T] : W)$, respectively, continuous with respect to x and (x, y), respectively, such that $\mathbf{P}_x$ is the probability measure of the X-process starting from x at time $t = 0$, and $\mathbf{P}^T_{x,y}$ is the probability measure for the X-process that starts from x at time $t = 0$ and is conditioned to pass through y at time $t = T$.

Feynman-Kac formula for matrix valued Hamiltonians over a local field

Theorem (Feynman-Kac)

The operator e^{-tH_b}, $(t > 0)$, is an integral operator on $L^2(W)$ with kernel $K_{t,b}$ on $W \times W$ which is represented by the following integral on the space $\mathcal{D}_t = \mathcal{D}([0, t] : W)$:

$$K_{t,b}(x : y) = \int_{\omega \in \mathcal{D}_t} E(-V \circ \omega)(t)\, dP^t_{x,y}(\omega) \cdot p_{t,b}(x - y) \qquad (x, y \in W)$$

where E is the time-ordered exponential mentioned earlier.

Proof of Feynman-Kac formula (sketch)

Proof (sketch).

Defining a family of operators $(\pi_t)_{t>0}$ on $L^2(W)$ by

$$(\pi_t f)(x) = \int_W \left\{ \int_{\mathcal{D}_t} E(-V \circ \omega)(t) f(\omega(t)) \, dP^t_{x,y} \right\} p_{t,b}(x-y) \, dy$$
$$= \int_W \left\{ \int_{\mathcal{D}_t} E(-V \circ \omega)(t) \, dP^t_{x,y} \right\} p_{t,b}(x-y) f(y) \, dy \,,$$

we prove that the $(\pi_t)_{t>0}$ form a strongly continuous semigroup, and that the infinitesimal generator of $(\pi_t)_{t>0}$ is an extension of the operator H_b. The theorem then follows from Hille-Yosida theory. □

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To appear in *Journal of Mathematical Analysis and Applications*.